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ON THE ORDER OF APPROXIMATION IN APPROXIMATIVE TRIADIC DECOMPOSITIONS OF TENSORS

Thomas LEHMKUHL and Thomas LICKTEIG

Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, D-7400 Tübingen, Fed. Rep. Germany

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Abstract. Based on Alder's (1984) result on the equivalence of algebraic and topological border rank of tensors over algebraically closed ground fields, we give an upper bound for the order of approximation in approximative triadic decompositions. We also treat the analogous question for real closed fields. In particular we carry over Alder's result to the case of real closed fields.

1. Introduction

Let k be a field. The *rank* $R(t)$ of a tensor $t\in k^n\otimes k^m\otimes k^l$ is defined as the minimum number $r\in\mathbb{N}$ such that t has a decomposition into triads

$$t=\sum_{\rho=1}^{r} u_\rho\otimes v_\rho\otimes w_\rho$$

where $u_\rho\in k^n$, $v_\rho\in k^m$, $w_\rho\in k^l$. Equivalently we have in coordinates (with respect to the canonical bases)

$$t_{ijk}=\sum_{\rho=1}^{r} u_{\rho i}\cdot v_{\rho j}\cdot w_{\rho k}\quad (i\leq n, j\leq m, k\leq l).$$

This notion is intimately related to the computational complexity of bilinear forms, and has been introduced by Strassen [19] (for motivation and background see also Borodin-Munro [4]; as a general reference on bilinear computational complexity, also for readers outside the field of algebraic complexity theory, we recommend De Groote's book [8] as an excellent introduction). The most prominent bilinear problem in fact is matrix multiplication, for which, by the work by Bini [2], Bini-Capovani-Lotti-Romani [3], Schönhage [17], Pan [13], Romani [15], Coppersmith-Winograd [5], Strassen [20] and Coppersmith-Winograd [6], improved upper bounds on the exponent of matrix multiplication were found using a concept of "approximative algorithms". Following Schönhage's [17] terminology (cf. [8, Chapter II 6]) we define the *approximative rank of order h*, $R_h(t)$, of some t as the minimum number $r \in \mathbb{N}$ such that an identity

$$(*) \qquad \sum_{\rho=1}^{r} u_{\rho i} \cdot v_{\rho j} \cdot w_{\rho k} = \varepsilon^h t_{ijk} + \varepsilon^{h+1} \cdot q_{ijk} \quad (i \le n, j \le m, k \le l)$$

with polynomials $u_{\rho i}, v_{\rho j}, w_{\rho k}, q_{ijk} \in k[\varepsilon]$ is possible. Clearly, $R(t) = R_0(t) \geq R_1(t) \geq \cdots$ (multiply by ε in (*)), and the minimum is called the *algebraic border rank* of t, $\underline{R}(t)$. (For the significance of this notion and its relation to matrix multiplication see [8].)

In this paper we investigate the question of how large h must be chosen such that $R_h(t) = \underline{R}(t)$. Assuming the ground field to be algebraically closed or real closed, we shall give an upper bound function for the maximally necessary h in terms of r and the format (n, m, l), independent of the specific tensor t. This is motivated by the problem of effectively computing border rank. (It is immediate that for fixed h, the degrees of $u_{\rho i}, v_{\rho i}, w_{\rho k}$ in (*) can be assumed to be bounded by h, and then (*) may be equated in a system of polynomial equations.)

The proof of the upper bound employs the topological approach by Alder [1]. If $k^n \otimes k^m \otimes k^l$ carries a topology, one can define the *topological border rank* $\underset{\sim}{R}(t)$ of t as the minimum $r \in \mathbb{N}$ such that t is contained in the closure of $\{t: R(t) \le r\}$. In the sequel we consider algebraically closed and real closed ground fields k. In the algebraically closed case, $k^n \otimes k^m \otimes k^l \cong k^{nml}$, as an affine space in the sense of algebraic geometry, carries in a natural way the Zariski topology, and in the real closed case, it carries the topology coming from the absolute value on k.

As shown by Alder [1], $\underline{R} = \underset{\sim}{R}$ for algebraically closed k (see also [20, Theorem 5.8]). Based on his approach we give a quantitative analysis in the subsequent theorem, also for the case of real closed fields.

Theorem. *For $t \in k^n \otimes k^m \otimes k^l$ the following statements are equivalent*:

(i) $\underset{\sim}{R}(t) \le r$,

(ii) $\underline{R}(t) \le r$,

(iii) $R_h(t) \le r$

where

$$h = \left(\frac{(n+m+l-3)!}{(n-1)!(m-1)!(l-1)!} \right)^r \le 3^{(n+m+l-3)r}$$

if k is algebraically closed, and

$$h = 3((n+m+l-2)r + 10)^{2^{(n+m+l-2)r-1}-1}$$

if k is real closed.

An immediate consequence is Alder's result for real closed fields.

Corollary. *For real closed fields, algebraic and topologic border rank coincide*: $\underline{R} = \underset{\sim}{R}$.

2. Preliminaries

Our proof of the Theorem will require some results from classical algebraic geometry, as can be found in [9, 18]. In order to make the paper understandable for readers with less prior knowledge of algebraic geometry, in this section we collect some facts and definitions to be used later. Also proofs are worked out fairly explicitly.

Let k be algebraically closed. By an affine variety we mean an irreducible algebraic set in some k^n. Hence we deal with embedded varieties. Similarly, projective varieties are supposed to be irreducible and embedded in some $\mathbb{P}_k^n$. We then denote by

- $A(V)$ the affine coordinate ring of an affine variety $V \subseteq k^n$ (polynomial functions on V),
- $I(V)$ its ideal in $k[X_1, \ldots, X_n] = A(k^n)$ (functions zero on V),
- $S(V)$ the homogeneous coordinate ring of a projective variety $V \subseteq \mathbb{P}_k^n$,
- $I_+(V)$ its homogeneous ideal in $k[X_0, \ldots, X_n] = S(\mathbb{P}_k^n)$,
- $\mathcal{O}_{x,V}$ the local ring of a point x on V (affine or projective),
- $K(V)$ the function field of V,
- $\Sigma(V)$ the singular locus of V.

We recall the notion of the degree of a projective variety $V \subseteq \mathbb{P}_k^n$ which is defined as $(\dim V)!$ times the leading coefficient of the Hilbert polynomial of the homogeneous coordinate ring of V (see [9, p. 52]). For intersections with hypersurfaces $H \subseteq \mathbb{P}_k^n$, $V \not\subseteq H$, *Bezout's theorem* states

$$\sum i(V, H; W) \cdot \det W = \deg V \cdot \deg H$$

where the sum ranges over all irreducible components W of $V \cap H$, and $i(V, H; W)$ denotes the intersection multiplicity of V and H along W (see [9, p. 53] for a proof). If $V \subseteq k^n$ is an affine variety, its degree is defined as the degree of its projective closure in $\mathbb{P}_k^n$. In order to get degree bounds we shall apply in most cases the affine *Bezout inequality*:

$$\sum_{\substack{W \text{ component} \\ \text{of } U \cap V}} \deg W \leq \deg U \cdot \deg V,$$

valid for arbitrary affine varieties $U, V \subseteq k^n$. This can be deduced from the projective version (cf. [16]). For a direct proof we refer to Heintz [10].

In the real closed case of our theorem, we are concerned with reality questions. For that purpose we list some definitions and results on real algebraic geometry (cf.

[7, 11]). As we shall see, the statement of the theorem can be expressed by a first-order formula in the theory of real closed fields, so by the transfer principle (see [14, p. 51]), we may restrict considerations to the field $\mathbb{R}$ of real numbers.

In affine n-spaces $\mathbb{R}^n$ over $\mathbb{R}$ we shall use the topology induced by the euclidian distance and in projective n-spaces $\mathbb{P}^n_{\mathbb{R}}$ over $\mathbb{R}$, the quotient topology from $\mathbb{R}^{n+1}\backslash\{0\}$. Note that this topology induces the topology on $\mathbb{R}^n$ (via the usual embedding in $\mathbb{P}^n_{\mathbb{R}}$).

Let V be an affine variety in the complex n-space $\mathbb{C}^n$. V is called real defined if $I(V)$ is generated by polynomials in $\mathbb{R}[X_1,\dots,X_n]$. Set $V_{\mathbb{R}} := V\cap\mathbb{R}^n$. V is called real if it is real defined and has a smooth point lying in $V_{\mathbb{R}}$. A projective variety $V\subseteq\mathbb{P}^n_{\mathbb{C}}$ is called real if its affine cone is real. A prime ideal $\mathfrak{P}\subseteq\mathbb{R}[X_1,\dots,X_n]$ is called real if its zero set in $\mathbb{C}^n$ is a real variety. Real ideal $I(V_{\mathbb{R}})$, real coordinate ring $A(V_{\mathbb{R}})$, real function field $K(V_{\mathbb{R}})$ and real local ring $\mathcal{O}_{x,V_{\mathbb{R}}}$ $(x\in V_{\mathbb{R}})$ are defined in the obvious way: $I(V_{\mathbb{R}}) := I(V)\cap\mathbb{R}[X_1,\dots,X_n]$, $A(V_{\mathbb{R}}) := \mathbb{R}[X_1,\dots,X_n]/I(V_{\mathbb{R}})$, $K(V_{\mathbb{R}}) :=$ quotient field of $A(V_{\mathbb{R}})$ and $\mathcal{O}_{x,V_{\mathbb{R}}} := \mathcal{O}_{x,V}\cap K(V_{\mathbb{R}})$. Analogously for a real projective variety V, $I_+(V_{\mathbb{R}})$, $S(V_{\mathbb{R}})$ etc. Reality of V means that the real part $V_{\mathbb{R}}$ is representative for V, i.e. the Zariski closure of $V_{\mathbb{R}}$ in $\mathbb{C}^n$ is V. Thus we also call an irreducible algebraic set W in $\mathbb{R}^n$ a real variety, and denote its Zariski closure in $\mathbb{C}^n$ by $W^{\mathbb{C}}$. Its degree is defined as $\deg W := \deg W^{\mathbb{C}}$. $W^{\mathbb{C}}$ is real as defined above. By the Dubois-Risler Nullstellensatz (cf. [7]) a prime ideal $\mathfrak{P}\subseteq\mathbb{R}[X_1,\dots,X_n]$ is real iff the quotient field of $A=\mathbb{R}[X_1,\dots,X_n]/\mathfrak{P}$ is a real field, i.e. -1 is not a sum of squares. In this case the real zeros of $\mathfrak{P}$ are in one-to-one correspondence with maximal ideals $\mathfrak{M}$ of A for which $A/\mathfrak{M}$ is isomorphic to $\mathbb{R}$.

Let $V\subseteq k^n$, $W\subseteq k^m$ be affine varieties. A morphism $\varphi: V\to W$ is a mapping given by polynomials $f_1,\dots,f_m\in A(k^n)$. A morphism of projective varieties $\varphi: V\to W$, $V\subseteq\mathbb{P}^n_k$, $W\subseteq\mathbb{P}^m_k$ is a mapping locally given by homogeneous polynomials $f_0,\dots,f_m\in S(\mathbb{P}^n_k)$ all having the same degree (cf. [18]). A morphism φ is called dominant if $\varphi(V)$ is Zariski dense in W. If $k=\mathbb{C}$ and V and W are real varieties, φ is called a morphism of real varieties if the above polynomials can be chosen in $A(\mathbb{R}^n)$, respectively in $S(\mathbb{P}^n_{\mathbb{R}})$.

To prove the Theorem we shall consider the problem in the following slightly more general setting. Given affine varieties $V\subseteq k^n$, $W\subseteq k^m$, a morphism $\varphi: V\to W$ given by polynomials $f_1,\dots,f_m$ and a point $t=(t_1,\dots,t_m)\in W$ such that t is contained in the Zariski closure $\overline{\varphi(V)}$, respectively in the classical closure if $k=\mathbb{R}$, then one can construct a curve $C\subseteq V$ whose degree is bounded in terms of the degree of V and the degrees of the f's such that $t\in\overline{\varphi(C)}$ (Sections 3, 4). Finally, a "local uniformization" will give us formal Laurent series $p_1,\dots,p_n\in k((\varepsilon))$ with pole order bounded by the degree of C such that $f_i(p_1,\dots,p_n)\in k[[\varepsilon]]$, $f_i(p_1,\dots,p_n)_{(\varepsilon=0)}=t_i$ $(i=1,\dots,m)$ and $p_1,\dots,p_n$ satisfy the relations of C, i.e. for all $f\in I(C)$ we have $f(p_1,\dots,p_n)=0$ (Section 5). From this the statement of the Theorem follows easily (Section 6). In the algebraically closed case the curve C can be constructed by hyperplane sections (Section 3). This does not work in the real case. Here we use a minimization technique (Section 4). In both cases Bezout's theorem will yield the desired estimates on the degree of the curve.

3. Curve selection if k is algebraically closed

Proposition 1. *Let $V \subseteq k^n$, $W \subseteq k^m$ be affine varieties, $\varphi : V \to W$ dominant, $t \in W \setminus \varphi(V)$. Then there is a curve $C \subseteq V$ such that $t \in \overline{\varphi(C)}$ and* $\deg C \leqslant \deg \operatorname{graph} \varphi|_C \leqslant \deg \operatorname{graph} \varphi$.

Proof. We have the situation

$$\begin{tikzcd}
 & k^n \times k^m \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\cup"] & \\
k^n \arrow[d, phantom, "\cup"] & \operatorname{graph} \varphi \arrow[dr] & k^m \arrow[d, phantom, "\cup"] \\
V \arrow[ur] \arrow[rr, "\varphi"] & & W
\end{tikzcd}$$

where π_1, π_2 are the canonical projections. First we show

$$\dim W > 1 \Rightarrow \begin{cases} \exists \text{subvariety } V_1 \subsetneq V,\ \varphi|_{V_1} \text{ dominating some} \\ W_1 \subsetneq W \text{ of dimension } \dim W - 1 \text{ so that} \\ \deg \operatorname{graph} \varphi|_{V_1} \leqslant \deg \operatorname{graph} \varphi \text{ and } t \in \overline{\varphi(V_1)}. \end{cases}$$

By Chevalley's theorem the image of φ is a constructible subset of W (i.e. in the Boolean algebra generated by the Zariski open subsets of W) and contains an open dense part of W. Let $U \subseteq W$ be nonvoid, open, affine with $U \subseteq \operatorname{im} \varphi$, $U = W \setminus W_0$ for some $W_0 \subseteq W$. Choose a hyperplane $H \subseteq k^m$ through $t \in W_0 \subseteq W$ intersecting W_0 and thus W properly. Then $\operatorname{im} \varphi \cap H \neq \emptyset$. Choose an irreducible component W_1 of $W \cap H$ through t, and a component V_1 of $\varphi^{-1}(W_1)$ with $\overline{\varphi(V_1)} = W_1$. Then $\operatorname{graph} \varphi|_{V_1}$ is a component of the intersection $\operatorname{graph} \varphi \cap \pi_2^{-1} H$, thus $\deg \operatorname{graph} \varphi|_{V_1} \leqslant \deg \operatorname{graph} \varphi$, by Bezout's inequality.

So it remains to show

$$\dim W = 1, \dim V > 1 \Rightarrow \begin{cases} \exists \text{subvariety } V_1 \subsetneq V \text{ so that } \varphi|_{V_1} \\ \text{dominates } W \text{ and } \deg \operatorname{graph} \varphi|_{V_1} \\ \leqslant \deg \operatorname{graph} \varphi. \end{cases}$$

To see this, we choose $x \in W$ such that each component of $\varphi^{-1}(x)$ has dimension $\dim V - 1$ ($\geqslant 1$), and a hyperplane $H \subseteq k^n$, such that the intersection of H with $\varphi^{-1}(x)$ is proper and nonvoid. Thus H intersects also V properly. Let $H \cap V = V_1 \cup \cdots \cup V_s$ be the decomposition into irreducible components. Then $\varphi|_{V_j}$ dominates W for some j. For otherwise we had finitely many points $\varphi(V_j) = \{x_j\}$ with $x_{j_0} = x$ for some j_0. But then $\varphi^{-1}(x)$ contains V_{j_0} of dimension $\dim V - 1$, and $V_{j_0} \subseteq H$, contradicting the assumption that H intersects $\varphi^{-1}(x)$ properly. So say $\overline{\varphi(V_1)} = W$.

Then graph $\varphi|_{V_1} = \text{graph } \varphi \cap \pi_1^{-1}H$, thus $\deg \text{graph } \varphi|_{V_1} \leq \deg \text{graph } \varphi$, by Bezout. □

4. Curve selection if $k = \mathbb{R}$

Let $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$, $\varphi: V \to W$ be given by polynomials $f_1, \ldots, f_m \in \mathbb{R}[X_1, \ldots, X_n]$ and $t = (t_1, \ldots, t_m) \in W$. By the Tarski-Seidenberg theorem $\varphi(V)$ is a semialgebraic subset of W (i.e. in the Boolean algebra generated by the open subsets $\{t: f(t) > 0\} \subseteq W$ where f is a real polynomial). Here hyperplane sections do not work in general, as can be easily visualized in the example of the semialgebraic set $S = \{0 < x, y < 1, x^5 - y^2 < x^3\}$ in $\mathbb{R}^2$ and the point $(0, 0) \in \bar{S}$.

It is clear that $t \in \overline{\varphi(V)}$ iff $f = \sum_{i=1}^n (f_i - t_i)^2$ takes arbitrarily small values on V.

The following analogue to Proposition 1 has been inspired by Milnor [12, Section 3].

Proposition 2. *Let $V \subseteq \mathbb{R}^n$ be a real variety of dimension d, $f \in \mathbb{R}[X_1, \ldots, X_n]$ strictly positive on V with $0 \in \overline{f(V)}$. Then there is a real curve $C \subseteq V$ such that $0 \in \overline{f(C)}$ and*

$$\deg C \leq (\deg V)^{2^{d-1}} \cdot (n + \deg f + 4)^{2^{d-1}-1}.$$

Proof. It suffices to show

$$(1)\quad \dim V > 1 \Rightarrow \begin{cases} \exists \text{real subvariety } W_1 \subsetneq V \text{ such that } 0 \in \overline{f(W_1)} \text{ and} \\ \deg W_1 \leq [(\deg V)((n-d)(\deg V - 1) + \deg f + 4)]^{2^{d-1-\dim W_1}} \\ \qquad \leq [(\deg V)^2(n + \deg f + 4)]^{2^{d-1-\dim W_1}} \end{cases}$$

for then (by induction) we get a chain of subvarieties $V = W_0 \supsetneq W_1 \supsetneq \cdots \supsetneq W_k = C$ having degrees bounded according to

$$\deg W_i \leq [(\deg W_{i-1})^2(n + \deg f + 4)]^{2^{\dim W_{i-1} - 1 - \dim W_i}}$$
$$\leq (\deg V)^{2^{d - \dim W_i}}(n + \deg f + 4)^{2^{d - \dim W_{i-1}}} \quad (1 \leq i \leq k).$$

To prove (1), we choose coordinates $X_1, \ldots, X_n$ on $\mathbb{R}^n$ such that the following properties are satisfied:

(2a) V lies in none of the coordinate hyperplanes,

(2b) the restrictions on V of each projection onto a d-dimensional coordinate plane is dominant.

We define rational functions $f_0, \ldots, f_n$ on V by $f_0 := f$ and $f_i := (1/g_i + 1)f$ for $1 \leq i \leq n$ where $g_i := X_i^2 + 1$. By construction of f_i

$$(3)\quad 0 \in \overline{f_i(W)} \Leftrightarrow 0 \in \overline{f(W)} \quad \text{for any } W \subseteq V.$$

Let $S_\rho = \{x \in \mathbb{R}^n: r(x) = \rho^2\}$ denote the sphere of radius ρ, where $r(X) := \sum_{i=1}^n X_i^2$. Let $I(V) = (h_1, \ldots, h_s)$. We define for $0 \leq i \leq n$

$$V_i := \{x \in V: \operatorname{rank}(\mathrm{d}f_i(x), \mathrm{d}r(x), \mathrm{d}h_1(x), \ldots, \mathrm{d}h_s(x)) \leq n - d + 1\} \supseteq \Sigma(V).$$

Observe that these are algebraic sets, for the differential df_i may be equivalently replaced by $g_i^2 \cdot df_i$.

The proof of (1) runs along the subsequent four statements which by (3) clearly imply (1):

$$(4) \qquad 0 \in \overline{f_i(V_i)} \quad \text{for } 0 \leq i \leq n,$$

$$(5) \qquad V_i \subsetneq V \quad \text{for some } i,$$

$$(6) \qquad V_i \subsetneq V \Rightarrow \begin{cases} \exists g \notin I(V), \deg g \leq (n-d)(\deg V - 1) + \deg f + 4 \\ \text{such that } V_i \subseteq \{g = 0\}, \end{cases}$$

$$(7) \qquad \left.\begin{matrix} W \subsetneq V \text{ real subvariety,} \\ W \subseteq \{g=0\} \not\supseteq V \end{matrix}\right\} \Rightarrow \begin{cases} \exists \text{real subvariety } W_1, \\ \deg W_1 \leq [(\deg V)(\deg g)]^{2^{d-1-\dim W_1}} \\ \text{such that } W \subsetneq W_1 \subsetneq V. \end{cases}$$

(4): This is clear if $0 \in \overline{f_i(\Sigma(V))}$. If not, we consider the d-dimensional real analytic manifold $M := V \backslash \Sigma(V)$. r has only finitely many critical values on M (i.e. values $r(x)$ for critical points $x \in M$, i.e. $dr(x) = 0$ on the tangent space $T_{x,M}$). Thus for $\rho \geq \rho_0$ (some ρ_0), $M_\rho := M \cap S_\rho$ is a real analytic manifold of dimension $d-1$ (since M has only finitely many connected components, on one of which f_i takes arbitrarily small values, this component is unbounded (otherwise f_i and thus f had a zero on V), so $M_\rho \neq \emptyset$ for $\rho \geq \rho_0$). In particular $dr(x) \neq 0$ on $T_{x,M}$ for $\rho \geq \rho_0$. Let x' be a minimum of f_i on the compact set $V \cap S_\rho = M_\rho \cup (\Sigma(V) \cap S_\rho)$. If $x' \in \Sigma(V) \cap S_\rho$, then $x' \in V_i$. Else $x' \in M_\rho$, and x' is a critical point of $f_i|_{M_\rho}$, thus $df_i(x') = 0$ on T_{x',M_ρ}, and thus $x' \in V_i$. It is now clear that $0 \in \overline{f_i(V_i)}$, for $f_i(x_\nu) \to 0$ for a sequence $x_\nu \in V$ implies $f_i(x'_\nu) \to 0$ where x'_ν is a minimum of $f_i|_{V \cap S_{\|x_\nu\|}}$.

(5): Contrary to the assertion we assume $V = V_0 = V_1 = \cdots = V_n$ and choose a smooth point $x \in V$ having only nonzero coordinates such that $dr(x) \neq 0$ on $T_{x,V}$ (use (2a)). Then $V_0 = V_1 = \cdots = V_n$ implies $df_i(x) \in \operatorname{span}(dr(x), dh_1(x), \ldots, dh_s(x))$ for $0 \leq i \leq n$. From $df_i - (1/g_i + 1)\, df = f\, d(1/g_i)$ $(i \geq 1)$ and

$$\operatorname{rank}\left(d\frac{1}{g_1}(x), \ldots, d\frac{1}{g_n}(x) \right) = n$$

it now follows that $d = \dim V = \dim T_{x,V} \leq 1$, contradicting $d > 1$.

(6): Let $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{R}^n$. We project V on the coordinate planes $E_j = \operatorname{span}(e_1, \ldots, e_d, e_j)$ $(d+1 \leq j \leq n)$. By (2b) the Zariski closures of the images are real hypersurfaces $H_j \subseteq E_j$ with $\deg H_j \leq \deg V$. The generators h'_j of their ideals $I(H_j) = \mathbb{R}[X_1, \ldots, X_d, X_j] \cap I(V)$ are polynomials with linearly independent differentials $dh'_j(x)$ for all $x \in V$ with $\prod_{j=d+1}^n ((\partial/\partial X_j) h'_j) \neq 0$ (by (2b) this product does not vanish everywhere on V). Thus, by our premise $V_i \subsetneq V$,

$$\operatorname{rank}(g_i^2(x) \cdot df_i(x), dr(x), dh'_{d+1}(x), \ldots, dh'_n(x)) = n - d + 2$$

for almost all $x \in V$, so

$$V_i \subseteq \{x \in V: \operatorname{rank}(g_i^2(x) \cdot df_i(x), dr(x), dh'_{d+1}(x), \ldots, dh'_n(x)) \leq n-d+1\} \subsetneq V.$$

Now, choosing a suitable minor and taking into account $\deg h'_j = \deg H_j \leq \deg V$ $(d+1 \leq j \leq n)$ we obtain the asserted g.

(7): By our premise, $V^{\mathbb{C}}$ and the hypersurface $H = \{x \in \mathbb{C}^n: g(x) = 0\}$ intersect properly. Let $X_1 \cup \cdots \cup X_k = V^{\mathbb{C}} \cap H$ be the decomposition into irreducible components, each X_ν having dimension $d-1$. By Bezout's inequality

$$\sum_\nu \deg X_\nu \leq (\deg V)(\deg g) =: A \quad \text{(for short)},$$

and $W^{\mathbb{C}} \subseteq X_1$, say. If X_1 is real, we take $W_1 = (X_1)_{\mathbb{R}}$. Otherwise we ask whether $X_1 = X_1^c$; here X_1^c denotes the variety consisting of all coordinatewise complex conjugate points of X_1. If $X_1 \neq X_1^c$, then $W^{\mathbb{C}} \subseteq X_1 \cap X_1^c$; else X_1 is real defined and then $W^{\mathbb{C}} \subseteq \Sigma(X_1)$. Thus in both cases, by the lemma below and $\deg X_1 = \deg X_1^c$, $W^{\mathbb{C}}$ is contained in a proper subvariety $Y \subsetneq X_1$ of degree $\leq (\deg X_1)^2$, by Bezout. If Y is real, we are done; else we repeat the latter process until we get a real X, $W^{\mathbb{C}} \subseteq X \subsetneq V^{\mathbb{C}}$, with $\deg X \leq A^{2^{\dim X_1 - \dim X}} = A^{2^{d-1-\dim X}}$. Now take $W_1 = X_{\mathbb{R}}$. $\square$

Lemma. *Let $V \subseteq k^n$ be an affine variety, k algebraically closed. Then there is a hypersurface H not containing V of degree $\leq \deg V - 1$ such that $\Sigma(V) \subseteq V \cap H$.*

Proof. Choose a projection $\pi: k^n \to E$ onto a $(\dim V + 1)$-dimensional affine subspace $E \subseteq k^n$, which restricts to a finite birational morphism $\pi': V \to W$, onto a hypersurface $W \subseteq E$ of degree $\leq \deg V$. Since π' is finite, $\pi'(\Sigma(V)) \subseteq \Sigma(W)$. One of the derivatives of the generators of $I(W)$ yields a hypersurface $H' \subseteq E$ cutting W. Now take $H = \pi^{-1}H'$. $\square$

5. Local uniformization

The aim will now be to find local parametrizations of the curves just constructed in a point "at infinity" by means of formal Laurent series of bounded pole order. For the readers's convenience we first recall some concepts from algebraic geometry.

Normalization of projective curves. Let k be an algebraically closed field and $C \subseteq k^n$ an affine curve with affine coordinate ring $A(C)$. The integral closure $\overline{A(C)}$ of $A(C)$ in $K(C)$ is the affine coordinate ring of a normal affine curve $\tilde{C}$. $\tilde{C}$ is a smooth curve, and there is a canonical finite birational mapping $\pi: \tilde{C} \to C$. $\tilde{C}$ together with the mapping π is called the normalization of C. If $C \subseteq \mathbb{P}_k^n$ is a projective curve and $\mathbb{P}_k^n = \bigcup U_i$ is an affine covering of $\mathbb{P}_k^n$ then by gluing together the normalizations $\tilde{C}_i$ of $U_i \cap C$ one obtains a projective normal (and thus smooth) curve $\tilde{C}$ and a finite birational mapping $\pi: \tilde{C} \to C$, called the normalization of C (for details see [18, p. 120f]). Moreover, $\tilde{C}$ is unique up to canonical isomorphisms.

If $k = \mathbb{C}$ and C is a real affine curve, then by the Dubois-Risler Nullstellensatz $K(C_{\mathbb{R}})$ is a formal real field, and hence the integral closure $\overline{A(C_{\mathbb{R}})}$ of $A(C_R)$ is the real coordinate ring of a real curve $\tilde{C}$ which is just the normalization of C, as its affine coordinate ring $\overline{A(C)}$ is nothing but the scalar extension $\mathbb{C} \otimes_{\mathbb{R}} \overline{A(C_R)}$. Similarly, the normalization of a real projective curve is real, which follows from the explicit construction of $\tilde{C}$ from its affine parts given in [18].

Localization. Let S be a graded ring and $T \subseteq S$ a multiplicative subset consisting of homogeneous elements. Then the localization S_T has a natural grading given by $\deg(f/g) = \deg f - \deg g$ for homogeneous $f \in S$ and $g \in T$. The subring of S_T of elements of degree 0 is denoted by $S_{(T)}$. If $\mathfrak{P} \subseteq S$ is a homogeneous prime ideal, a standard notation is $S_{\mathfrak{P}}$ for S_T and $S_{(\mathfrak{P})}$ for $S_{(T)}$ where T is the set of homogeneous elements not in $\mathfrak{P}$. Let V be a projective variety, $x \in V$ and let $\mathfrak{M}_x$ denote the prime ideal generated by all homogeneous elements in $S(V)$ vanishing at x. Then $S(V)_{(\mathfrak{M}_x)} = \mathcal{O}_{x,V}$. If $f \in S(V)$ is a homogeneous element, then the subset $D_+(f) \subseteq V$ consisting of all points at which f does not vanish is (isomorphic to) an affine variety, and we have $A(D_+(f)) = S_{(f)}$. Here $S_{(f)}$ is the standard notation for $S_{(T)}$ where T is the multiplicative system of all powers of f.

Intersection (cf. [9, Chapter I, Section 7]). Let V be a projective variety in $\mathbb{P}_k^n$ over an algebraically closed field k defined by a homogeneous ideal $I := I_+(V) \subseteq S(\mathbb{P}_k^n)$, and H a hypersurface in $\mathbb{P}_k^n$ not containing V defined by a homogeneous polynomial $f \in S := S(\mathbb{P}_k^n)$. For an irreducible component W of $V \cap H$ the intersection multiplicity $i(V, H; W)$ of V and H along W is defined as the length of the $S_{\mathfrak{P}}$-module $(S/I + (f))_{\mathfrak{P}}$, where $\mathfrak{P}$ is the prime ideal in S generated by all homogeneous elements vanishing on W. We have the following lemma.

Lemma. $i(V, H; W) = \operatorname{length}_{S_{(\mathfrak{P})}} S_{(\mathfrak{P})}/I_{(\mathfrak{P})} + (u)$, *where* $u = f/g$ *with an arbitrary homogeneous* g *not in* $\mathfrak{P}$ *of the same degree as* f.

Proof. There exists a decreasing sequence of homogeneous ideals $S/I + (f) = I_0 \supsetneq \cdots \supsetneq I_r = (0)$ such that for $0 \le j < r$, I_j/I_{j+1} is isomorphic to $(S/\mathfrak{P}_j)(d_j)$ for some homogeneous prime ideal $\mathfrak{P}_j$ and some integer d_j (cf. [9, p. 50]). Here (d_j) denotes a d_j-shift of the degree. Hence $i(V, H; W)$ is exactly the number of times $\mathfrak{P}$ occurs in the sequence $\mathfrak{P}_0, \ldots, \mathfrak{P}_{r-1}$. On the other hand, if $T \subseteq S$ denotes the set of homogeneous elements not in $\mathfrak{P}$, then one checks easily that $(S/\mathfrak{P}_j)(d_j)_T = 0$ if $\mathfrak{P}_j \neq \mathfrak{P}$, therefore $(S/\mathfrak{P}_j)(d_j)_{(\mathfrak{P})} = 0$ in this case, and $\operatorname{length}_{S_{(\mathfrak{P})}}(S/\mathfrak{P}_j)(d_j)_{(\mathfrak{P})} = 1$ if $\mathfrak{P}_j = \mathfrak{P}$. Now the assertion follows by considering the sequence of $S_{(\mathfrak{P})}$-modules

$$(S/I + (f))_{(\mathfrak{P})} \supseteq (I_1)_{(\mathfrak{P})} \supseteq \cdots \supseteq (I_r)_{(\mathfrak{P})} = (0). \qquad \square$$

Now we consider a projective curve $C \subseteq \mathbb{P}_k^n$ and its normalization $\tilde{C}$. Recall that $\tilde{C}$ is nonsingular, so for each $x \in \tilde{C}$ the local ring $\mathcal{O}_{x,\tilde{C}}$ of $\tilde{C}$ in x is a discrete valuation ring with residue field k. Thus its completion is isomorphic to the ring of formal power series in one variable, $\hat{\mathcal{O}}_{x,\tilde{C}} = k[[\varepsilon]]$. The projection $\pi : \tilde{C} \to C$ induces an embedding of $\mathcal{O}_{\pi(x),C}$ in $\mathcal{O}_{x,\tilde{C}}$, hence in $k[[\varepsilon]]$, i.e. each function regular in $\pi(x)$ is represented by a formal power series. Hence each rational function $f \in k(C)$ is represented by a Laurent series $p(f) \in k((\varepsilon))$.

Proposition 3. *Let $\pi : \tilde{C} \to C$ be the normalization of a projective curve $C \subseteq \mathbb{P}_k^n$ over an algebraically closed field k, $x_0 \in \tilde{C}$, $y_0 = \pi(x_0) \in C$ and let $f, g \in S(\mathbb{P}_k^n)$ be linear homogeneous polynomials nonzero on C. Assume $f(y_0) = 0$ and $g(y_0) \neq 0$. Then f/g restricted to C is regular in y_0, and for the order of its associated series $p(f/g) \in \mathcal{O}_{x_0,\tilde{C}} = k[[\varepsilon]]$ the following inequalities are valid:*

$$\operatorname{ord} p(f/g) \leq i(C, H; y_0) \leq \deg C,$$

where H is the hyperplane described by f. Moreover for $k = \mathbb{C}$, C a real projective curve, $f, g \in S(\mathbb{P}_{\mathbb{R}}^n)$ and a real point $x_0 \in \tilde{C}$, the parameter ε can be chosen such that the associated series has real coefficients.

Proof. There exists a linear form g' in $S(\mathbb{P}_k^n)$ such that $S(C)$ is integral over $k[f, g']$ [21, p. 200]. We first consider the case $g = g'$. By an integralness argument it follows that the hyperplane described by g does not intersect $C \cap H$. We set $u := f/g$, $A = S(C)_{(g)}$ and $B :=$ integral closure of A in its quotient field. A is the coordinate ring of the affine piece $D_+(g)$ of C containing $C \cap H$, and B is the one of its normalization. Both are integral over $k[u]$ and have the same quotient field $K(C)$. Since A and B are torsion-free finite modules over the principal ideal domain $k[u]$, they are free [11, p. 533], and by passing to quotients one sees that both have the same rank $d := [K(C) : k(u)]$. By definition $C \cap H$ consists exactly of the points lying over the point $u = 0$. Hence we have a one-to-one correspondence between the maximal ideals in A respectively in B over (u) and points of $C \cap H$ respectively $\pi^{-1}(C \cap H)$. More precisely we have primary decompositions

$$A \cdot u = \prod_{y \in C \cap H} \mathfrak{Q}_y,$$

$$B \cdot u = \prod_{x \in \pi^{-1}(C \cap H)} \tilde{\mathfrak{Q}}_x = \prod_{y \in C \cap H} \prod_{x \in \pi^{-1}(y)} \tilde{\mathfrak{Q}}_x$$

where $\mathfrak{Q}_y$ respectively $\tilde{\mathfrak{Q}}_x$ are primary with respect to $\mathfrak{M}_y$ respectively $\mathfrak{M}_x$, the maximal ideals belonging to y respectively x. Moreover, $B\mathfrak{Q}_y = \prod_{x \in \pi^{-1}(y)} \tilde{\mathfrak{Q}}_x$. Completing with respect to $(u) \subseteq k[u]$ we get an injection $\iota : \hat{A} \hookrightarrow \hat{B}$. Using the Chinese Remainder Theorem and the fact that there is an exponent e such that $\mathfrak{M}_y^e \subseteq \mathfrak{Q}_y \subseteq \mathfrak{M}_y$ and $\mathfrak{M}_x^e \subseteq \tilde{\mathfrak{Q}}_x \subseteq \mathfrak{M}_x$ for all x, y, we can rewrite as a product:

$$\hat{A} = \varprojlim_n A/(Au)^n = \varprojlim_n A \Big/ \Big(\prod_{y \in C \cap H} \mathfrak{Q}_y^n \Big) = \varprojlim_n \prod_{y \in C \cap H} A/\mathfrak{Q}_y^n$$

$$= \prod_{y \in C \cap H} \varprojlim_n A/\mathfrak{Q}_y^n = \prod_{y \in C \cap H} \varprojlim_n A/\mathfrak{M}_y^n = \prod_{y \in C \cap H} \hat{A}_{\mathfrak{M}_y}$$

where $\hat{A}_{\mathfrak{M}_y}$ denotes the completion of A with respect to $\mathfrak{M}_y$. Analogously

$$\hat{B} = \prod_{x \in \pi^{-1}(C \cap H)} \hat{B}_{\mathfrak{M}_x} = \prod_{y \in C \cap H} \prod_{x \in \pi^{-1}(y)} \hat{B}_{\mathfrak{M}_x}.$$

ι injects $\hat{A}_{\mathfrak{M}_y}$ into $\prod_{x \in \pi^{-1}(y)} \hat{B}_{\mathfrak{M}_x}$, and by the same argument this is the injection of the completion of A into the one of B with respect to $\mathfrak{M}_y$. Both are free $k[[u]]$-modules (as submodules of free modules; cf. [11, p. 532]) and—taking the sum of the ranks over all y—have the same rank. We have

$$\operatorname{rank}_{k[[u]]} \hat{A}_{\mathfrak{M}_y} = \dim_k \hat{A}_{\mathfrak{M}_y} / \hat{A}_{\mathfrak{M}_y} \cdot u.$$

As local Artinian rings are complete,

$$A_{\mathfrak{M}_y} / A_{\mathfrak{M}_y} \cdot u \cong \hat{A}_{\mathfrak{M}_y} / \hat{A}_{\mathfrak{M}_y} \cdot u.$$

Considering a maximal chain $A_{\mathfrak{M}_y} / A_{\mathfrak{M}_y} \cdot u = \mathfrak{A}_0 \supsetneq \cdots \supsetneq \mathfrak{A}_n = (0)$ of ideals we conclude from $\mathfrak{A}_i / \mathfrak{A}_{i+1} \cong A_{\mathfrak{M}_y} / A_{\mathfrak{M}_y} \cdot \mathfrak{M}_y \cong k$ that

$$\operatorname{rank}_{k[[u]]} \hat{A}_{\mathfrak{M}_y} = \operatorname{length}_{A_{\mathfrak{M}_y}} A_{\mathfrak{M}_y} / A_{\mathfrak{M}_y} \cdot u.$$

By the lemma the latter is just the intersection multiplicity $i(C, H; y)$. Applying Bézout's theorem we have proved

$$d = \sum_{y \in C \cap H} i(C, H; y) = \deg C.$$

Because $\hat{B}_{\mathfrak{M}_x} = \hat{\mathcal{O}}_{x, \tilde{C}} = k[[\varepsilon]]$, it follows that

$$\operatorname{rank}_{k[[u]]} \hat{B}_{\mathfrak{M}_x} = \dim_k \hat{B}_{\mathfrak{M}_x} / \hat{B}_{\mathfrak{M}_x} \cdot u$$

is exactly the order of the series associated to u in the point x. Since x_0 is one point among the x's considered above, we get from the equality of ranks

$$\deg C \geq i(C, H; y_0) = \sum_{x \in \pi^{-1}(y_0)} \dim_k \hat{B}_{\mathfrak{M}_x} / \hat{B}_{\mathfrak{M}_x} \cdot u$$
$$\geq \dim_k \hat{B}_{\mathfrak{M}_{x_0}} / \hat{B}_{\mathfrak{M}_{x_0}} \cdot u,$$

hence the assertion.

Now we drop the assumption $g = g'$ from the beginning, observing that g/g' is a unit in $\mathcal{O}_{y_0, C}$, so f/g and f/g' have in x_0 power series of the same order.

For the second part, we look at the real local ring $\mathcal{O}_{y_0, C_{\mathbb{R}}}$ which is a subring of $\mathcal{O}_{x_0, \tilde{C}_{\mathbb{R}}}$ which is isomorphic to $\mathbb{R}[[\varepsilon]]$ as x_0 is a real point of $\tilde{C}$. □

Corollary. (a) *Let k be algebraically closed, $C \subseteq k^n$ an affine curve and $\varphi : C \to k^m$ a morphism. Let $t = (t_1, \ldots, t_m) \in D := \overline{\varphi(C)}$. Then there exist series $p_1, \ldots, p_n \in k((\varepsilon))$ such that*

(i) $\operatorname{ord} p_1, \ldots, \operatorname{ord} p_n \geq -\deg C$,

(ii) *$p_1, \ldots, p_n$ satisfy the relations of C, i.e. for all $f \in I(C)$ we have $f(p_1, \ldots, p_n) = 0$,*

(iii) *$f_i(p_1, \ldots, p_n) \in k[[\varepsilon]]$ and $f_i(p_1, \ldots, p_n)_{(\varepsilon=0)} = t_i$ for $i = 1, \ldots, m$, where $f_1, \ldots, f_m$ are polynomials describing φ.*

(b) *If $k=\mathbb{C}$, C a real curve, $\varphi: C\to\mathbb{C}^m$ a morphism of real varieties and $t\in\overline{\varphi(C_{\mathbb{R}})}\subseteq\mathbb{R}^m$, then there exist series $p_1,\dots,p_n\in\mathbb{R}((\varepsilon))$ which have the properties* (i) *to* (iii) *above.*

Proof. In case (a) we can assume that D is not a single point. Then D is an affine curve. Let $\bar{C}$, $\bar{D}$ be the projective closures of C in $\mathbb{P}_k^n$ respectively of D in $\mathbb{P}_k^m$. φ induces a surjective map $\tilde{\varphi}$ from the normalization $\tilde{C}$ of $\bar{C}$ onto $\bar{D}$ [18, p. 92]. Let $x\in\tilde{C}$ be an inverse image of t. We consider the homogeneous coordinates $X_0,\dots,X_n$ on $\mathbb{P}_k^n$ and take as p_j the series corresponding to $X_j/X_0\in K(\bar{C})=K(\tilde{C})\subseteq k((\varepsilon))$. If X_j/X_0 is regular at x, then p_j has order $\geqslant 0$; if not, then X_0/X_j is regular at x and $1/p_j$ has order $\leqslant\deg C$ by Proposition 3. This proves (i) and (ii). But $f_i(p_1,\dots,p_n)$ is nothing else than the image of the ith coordinate function of k^m under the map of affine coordinate rings induced by $\tilde{\varphi}$ (which is finite) and therefore lies in $k[[\varepsilon]]$ and has value t_i for $\varepsilon=0$. This shows (iii).

In case (b) we start as in (a). As $\tilde{C}_{\mathbb{R}}$ is a classically closed subset of some real projective space $\mathbb{P}_{\mathbb{R}}^N$, it is compact, so $\tilde{\varphi}(\tilde{C}_{\mathbb{R}})$ is classically closed in $(\bar{D})_{\mathbb{R}}$. So we can find a real inverse image x. By the second part of Proposition 3 the series $p_1,\dots,p_n$ above have real coefficients. □

6. Proof of the Theorem

(iii) ⇒ (ii) ⇒ (i): This is clear.

(i) ⇒ (iii): We factor the rank decomposition $(u_\rho, v_\rho, w_\rho)_{\rho\leqslant r}\mapsto\sum_{\rho\leqslant r}u_\rho\otimes v_\rho\otimes w_\rho$

$$\begin{tikzcd} k^{(n+m+l)r} \arrow[rr, "\varphi"] \arrow[dr] & & k^n\otimes k^m\otimes k^l \\ & S^r \arrow[ur, "\psi"'] & \end{tikzcd}$$

where $S=\{u\otimes v\otimes w\}$ denotes the (affine) Segre variety. If k is algebraically closed, we look at ψ. ψ is given linearly, so

$$h:=\deg\operatorname{graph}\psi=\deg S^r=(\deg S)^r=\left(\frac{(n+m+l-3)!}{(n-1)!(m-1)!(l-1)!}\right)^r.$$

(For the latter equality consider the Hilbert polynomial of the projective Segre variety $\mathbb{P}(S)$

$$H(\mathbb{P}(S);d)=\binom{d+n-1}{n-1}\binom{d+m-1}{m-1}\binom{d+l-1}{l-1}$$

whose affine cone is just S.) For some given $t\in\overline{\operatorname{im}\psi}\setminus\operatorname{im}\psi$ we get, from Proposition 1 and the corollary to Proposition 3, formal Laurent series $s_{\rho ijk}\in k((\varepsilon))$, $\operatorname{ord}(s_{\rho ijk})\geqslant -h$, such that $(\sum_{\rho\leqslant r}s_{\rho ijk})_{(\varepsilon=0)}=t_{ijk}$ $(i\leqslant n, j\leqslant m, k\leqslant l)$. Choose $u_{\rho i}$, $v_{\rho j}$, $w_{\rho k}\in k((\varepsilon))$ such that $u_{\rho i}\cdot v_{\rho j}\cdot w_{\rho k}=s_{\rho ijk}$, so

$$\text{(8)}\qquad \left(\sum_{\rho=1}^{r}u_{\rho i}\cdot v_{\rho j}\cdot w_{\rho k}\right)_{(\varepsilon=0)}=t_{ijk}\quad (i\leqslant n, j\leqslant m, k\leqslant l).$$

By scaling we may assume w.l.o.g. that $\min_j \operatorname{ord} v_{\rho j} = \min_k \operatorname{ord} w_{\rho k} = 0$. Multiplying in (8) $u_{\rho i}$ by ε^h and taking h-jets we get the desired representation.

In the real closed case we get better bounds using φ. If $k = \mathbb{R}$ and $t \in \overline{\operatorname{im} \varphi} \setminus \operatorname{im} \varphi$, we consider $f = \sum_{i,j,k} (u_{\rho i} v_{\rho j} w_{\rho k} - t_{ijk})^2$ on $\mathbb{R}^{(n+m+l)r}$. We have $\deg f = 6$, and f is strictly positive, taking arbitrarily small values, also on open subsets $\{u_{\rho i} v_{\rho j} w_{\rho k} \neq 0 \colon \rho \leq r\}$. Thus, by scaling, we may restrict ourselves on an affine subspace $V \subset \mathbb{R}^{(n+m+l)r}$ of dimension $(n+m+l-2)r$. By Proposition 2 and the corollary to Proposition 3 we now conclude the existence of formal Laurent series $u_{\rho i}$, $v_{\rho j}$, $w_{\rho k} \in \mathbb{R}((\varepsilon))$, all having order

$$\geq -((n+m+l-2)r+10)^{2^{(n+m+l-2)r-1}-1} = -\tfrac{1}{3}h,$$

satisfying (8). As above, multiplying (8) by ε^h and taking h-jets we get the desired representation.

The general case of a real closed field k now follows from the transfer-principle (e.g. [14, p. 51, Corollary 5.8]). In fact, for fixed n, m, l, r and h (h as above), we may write down a sentence in the language of the elementary theory of real closed fields as

$$\forall t((\forall \delta > 0\, \exists \tilde{t} \colon R(\tilde{t}) \leq r \wedge (t_{ijk} - \tilde{t}_{ijk})^2 < \delta \text{ for all } i, j, k) \rightarrow R_h(t) \leq r),$$

which is true for $k = \mathbb{R}$, and hence for any real closed field k. $\square$

References

[1] A. Alder, Grenzrang und Grenzkomplexität aus algebraischer und topologischer Sicht, Dissertation, Univ. Zürich, 1984.

[2] D. Bini, Relations between EC-algorithms and APA-algorithms, applications, Nota interna B 79/8, I.E.J. Pisa, 1979.

[3] D. Bini, M. Capovani, G. Lotti and F. Romani, $O(n^{2.7799})$ complexity of matrix multiplication, *Inform. Process. Lett.* **8** (1979) 234-235.

[4] A. Borodin and I. Munro, *Computational Complexity of Algebraic and Numeric Problems* (American Elsevier, New York, 1975).

[5] D. Coppersmith and S. Winograd, On the asymptotic complexity of matrix multiplication, *SIAM J. Comput.* **11** (1982) 472-492.

[6] D. Coppersmith and S. Winograd, Matrix multiplication via Behrend's theorem, Technical Report RC 12104, Department of Math. Sciences, IBM Thomas J. Watson Research Center, Yorktown Heights; to appear in: *Proc. 19th Ann. ACM Symp. on Theory of Computing*, New York, 1987.

[7] G. Efroymson, Local reality on algebraic varieties, *J. Algebra* **29** (1974) 133-142.

[8] H.F. de Groote, *Lectures on the Complexity of Bilinear Problems*, Lecture Note in Computer Science, Vol. 245 (Springer, Berlin, 1986).

[9] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52 (Springer, Berlin, 1977).

[10] J. Heintz, Definability and fast quantifier elimination in algebraically closed fields, *Theoret. Comput. Sci.* **24** (1983) 239-277.

[11] S. Lang, *Algebra* (Addison-Wesley, Reading, MA, 1984).

[12] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Mathematics Studies, Vol. 61 (Princeton Univ. Press, 1968).

[13] V.Ya. Pan, New combinations of methods for the acceleration of matrix multiplication, *Comput. Math. Appl.* **7** (1981) 73-125.

[14] A. Prestel, *Lectures on Formally Real Fields*, Lecture Notes in Mathematics, Vol. 1093 (Springer, Berlin, 1984).
[15] F. Romani, Some properties of disjoint sums of tensors related to matrix multiplication, *SIAM J. Comput.* **11** (1982) 263-267.
[16] C.P. Schnorr, An extension of Strassen's degree bound, *SIAM J. Comput.* **10** (1981) 371-382.
[17] A. Schönhage, Partial and total matrix multiplication, *SIAM J. Comput.* **10** (1981) 434-455.
[18] J.R. Shafarevich, *Basic Algebraic Geometry*, Grundlehren, Band 213 (Springer, Berlin, 1974).
[19] V. Strassen, Vermeidung von Divisionen, *Crelles J. Reine Angew. Math.* **264** (1973) 184-202.
[20] V. Strassen, Relative bilinear complexity and matrix multiplication, *Crelles J. Reine Angew. Math.* **375/376** (1987) 406-443.
[21] O. Zariski and P. Samuel, *Commutative Algebra II*, Graduate Texts in Mathematics, Vol. 29, (Springer, Berlin, 1958).